Two-dimensional turbulence in the inverse cascade range

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Numerical and physical experiments on forced two-dimensional Navier-Stokes equations show that transverse velocity differences are described by "normal" Kolmogorov scaling $\langle(\Delta v)^{2n}\rangle\propto r^{2n/3}$ and obey Gaussian statistics. Since nontrivial scaling is a sign of the strong nonlinearity of the problem, these two results seem to contradict each other. A theory explaining these observations is presented in this paper. The derived self-consistent expression for the pressure gradient contributions leads to the conclusion that small-scale transverse velocity differences are governed by a linear Langevin-like equation, stirred by a nonlocal, universal, solution-dependent Gaussian random force. This explains the experimentally observed Gaussian statistics of transverse velocity differences and their Kolmogorov scaling. The solution for the PDF of longitudinal velocity differences is based on the numerical smallness of the energy flux in two-dimensional turbulence. The theory makes a few quantitative predictions that can be tested experimentally. [S1063-651X(99)13011-3]

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I. INTRODUCTION

Theoretical prediction of two inertial ranges, as a consequence of both energy and enstrophy conservation laws by the two-dimensional Euler equations, was and still is one of the most remarkable achievements of statistical hydrodynamics [1]. A direct, and the most important, outcome of these conservation laws is the fact that if a fluid is stirred by a random (or nonrandom) forcing, acting on a scale $l_f=1/k_f$, the energy produced is spent on creation of a large-scale ($l>l_f$) flow which cannot be dissipated in the limit of a large Reynolds number as $\nu\to 0$. This means that the dissipation terms are irrelevant in the inverse cascade range. Since the dissipation contributions are one of the most difficult obstacles on the road toward turbulence theory (see below), one can hope that in two dimensions the situation is greatly simplified. This hope is supported by recent numerical and physical experiments showing that as long as the integral scale $L_i\propto t^{3/2}$ is much smaller than the size of the system, the velocity field at scales $L_i\gg l\gg l_f$ is a stationary close-to-Gaussian process characterized by the structure functions

$$S_n=\overline{(u(x+r)-u(x))^n}\equiv\overline{(\Delta u)^n}\propto(Pr)^{n/3}, \tag{1}$$

where the pumping rate P is defined below [2–4]. Moreover, both numerical and physical experiments were not accurate enough to measure

$$s_{2n+1}=\frac{S_{2n+1}}{S_2^{(2n+1)/2}}\ll 1, \tag{2}$$

which were too small. This means that the observed probability density $P(\Delta u)$ was very close to being symmetric. This experimental fact differs from the outcome of the measurements in three dimensions, where s_n's are very large when n is not small. Thus the absence of strong (if any) intermittency in two-dimensional turbulence, and the proximity of the statistics of the velocity field to Gaussian, makes the problem seem tractable.

The equations of motion are (density $\rho\equiv 1$)

$$\partial_t v_i+v_j\partial_j v_i=-\partial_i p+\nu\nabla^2 v_i+f_i. \tag{3}$$

and

$$\partial_i v_i=0, \tag{4}$$

where $\mathbf{f}$ is a forcing function mimicking the large-scale turbulence production mechanism and, in a statistically steady state, the mean pumping rate $P=\overline{\mathbf{f}\cdot\mathbf{v}}$. In the inverse cascade range the dissipation terms in Eq. (3) will be irrelevant. Neglecting this and multiplying Eq. (3) by v_i, we readily obtain

$$E=\tfrac{1}{2}\overline{v^2}=Pt. \tag{5}$$

Thus in this case the energy grows linearly with time.

In this paper we define the force correlation function as

$$\langle f_i(\mathbf{k})f_j(\mathbf{k}')\rangle\propto P\left(\delta_{ij}-\frac{k_ik_j}{k^2}\right)\frac{\delta(k-k_f)}{k}\,\delta(\mathbf{k}+\mathbf{k}')\,\delta(t-t'), \tag{6}$$

so that

$$\overline{(f(x+r)-f(r))^2}\propto P(1-\cos(k_f r)). \tag{7}$$

It will be clear below that the forcing term enters the equations for the probability density of velocity differences exclusively through expression (7), and in the limit $k_f r\ll 1$ its contribution is $O((k_f r)^2)$ which is a well-known fact. In the energy cascading inertial range we are interested in this work, $k_f r\gg 1$, and the oscillating contribution can be ne-

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glected leading to the disappearence of the forcing scale from equation for the pair distribution function (PDF). Thus the general expression for the structure functions is

$$S_n(r)\propto(Pr)^{n/3}\left(\frac{r}{L_i(t)}\right)^{\delta_n}, \tag{8}$$

where the exponents δ_n denote possible deviations from the Kolmogorov scaling. If a statistically steady state exist in the limit $L_i \gg l \gg l_f$, then all $\delta_n=0$ since $L_i\propto t^{3/2}$. This would be proof of ''normal'' (Kolmogorov) scaling in the inverse cascade range, provided one can show that the PDF $P(\Delta u)$ in the inertial range is independent of its counterpart in the interval $l\approx l_f$. This is the subject of the present paper, which is organized as follows. In Sec. II the equations for the generating functions are introduced. Section III is devoted to a short analysis of the Polyakov theory of Burgers turbulence, some aspects of which are used in this paper. Some physical considerations, which are basic for the developing theory, are presented in Sec. IV. In Secs. V and VI the equations for the transverse and longitudinal probability density functions are derived and solved. A summary and discussion are presented in Sec. VII.

Now we would like to recall some well-known properties of velocity correlation functions in incompressible fluids, needed below. Consider two points $\mathbf{x}$ and $\mathbf{x}'$, and define $\mathbf{r}=\mathbf{x}-\mathbf{x}'$. Assuming that the x axis is parallel to the displacement vector $\mathbf{r}$, one can find that, in the two-dimensional flow $d=2$ for the separation r in the inertial range [5–7],

$$\frac{1}{r^{d+1}}\partial_r r^{d+1}S_3=\frac{12}{d}P, \tag{9}$$

where the pumping power $P=O(1)$ is a constant. In what follows we will often set $P=1$ and restore correct dimensionality at the end of calculations. Integrating Eq. (9) gives

$$S_3=\overline{(\Delta u)^3}\equiv\overline{(u(x')-u(x))^3}\equiv\frac{12}{d(d+2)}Pr \tag{10}$$

and

$$S_3^t=\overline{(\Delta v)^3}\equiv\overline{(v(x')-v(x))^3}=0, \tag{11}$$

where u and v are the components of velocity field parallel and perpendicular to the x axis (vector $\mathbf{r}$). Relations (9)–(11) resulting from equations of motion (3) are dynamic properties of the velocity field. The kinematics also gives something interesting:

$$\frac{1}{r^{d-2}}\frac{d}{dr}r^{d-1}S_2=(d-1)S_2^t\equiv(d-1)\overline{(\Delta v)^2}, \tag{12}$$

and in two dimensions we have

$$S_{3t}\equiv\overline{\Delta u(\Delta v)^2}=\frac{r}{3}\frac{d}{dr}S_3. \tag{13}$$

II. EQUATION FOR GENERATING FUNCTION

We consider the N-point generating function

$$Z=\langle e^{\lambda_i\cdot \mathbf{v}(x_i)}\rangle, \tag{14}$$

where the vector $\mathbf{x}_i$ define the positions of the points denoted $1\leqslant i\leqslant N$, and summation over the positions of the points x_j is assumed. Using the incompressibility condition, the equation for Z can be written

$$\frac{\partial Z}{\partial t}+\frac{\partial^2 Z}{\partial\lambda_{i,\mu}\partial x_{i,\mu}}=I_f+I_p, \tag{15}$$

with

$$I_f=\sum_j\langle\boldsymbol{\lambda}_\mathbf{j}\cdot f(\mathbf{x}_\mathbf{j})e^{\lambda_i u(x_i)}\rangle, \tag{16}$$

$$I_p=-\sum_j\lambda_j\left\langle e^{\lambda_i u(x_i)}\frac{\partial p(x_j)}{\partial x_j}\right\rangle. \tag{17}$$

The dissipation contributions have been neglected here as irrelevant.

In what follows we will be mainly interested in the probability density function of the two-point velocity differences which is obtained from Eqs. (14)–(17), setting $\lambda_\mathbf{1}+\lambda_\mathbf{2}=\mathbf{0}$ (see Ref. [8], and the theory developed below), so that

$$Z=\langle\exp(\boldsymbol{\lambda}\cdot\mathbf{U})\rangle, \tag{18}$$

where

$$\mathbf{U}=\mathbf{u}(\mathbf{x}')-\mathbf{u}(\mathbf{x})\equiv\Delta\mathbf{u}. \tag{19}$$

It is easy to see that in the incompressible case the equation for generating the function of velocity differences (18) is

$$\frac{\partial Z}{\partial t}+\frac{\partial^2 Z}{\partial\lambda_\mu\partial r_\mu}=I_f+I_p,$$

with

$$I_f=\langle\boldsymbol{\lambda}\cdot\boldsymbol{\Delta}\mathbf{f}e^{\boldsymbol{\lambda}\cdot\boldsymbol{\Delta}\mathbf{m}}\rangle$$

and

$$I_p=-\lambda\left\langle e^{\boldsymbol{\lambda}\cdot\boldsymbol{\Delta}\mathbf{u}}\left(\frac{\partial p(x_2)}{\partial x_2}-\frac{\partial p(x_1)}{\partial x_1}\right)\right\rangle.$$

The most interesting feature of these equations is the fact that the advective contributions are represented there in a closed form. To completely close the problem the expression for I_p is needed. The moments of the two-point velocity differences which in homogeneous and isotropic turbulence can depend only on the absolute values of two vectors [velocity difference $\mathbf{v}(\mathbf{x}')-\mathbf{v}(\mathbf{x})$ and displacement $\mathbf{r}\equiv\mathbf{x}'-\mathbf{x}$] and the angle θ between them with $\theta=\pi/2$ and $\theta=0$ correspond to transverse and longitudinal structure functions, respectively. It is easy to show [5,6] that the general form of the second-order structure function in the inertial range is

$$S_2(r,\theta)=\frac{2+\xi_2}{2}D_{LL}(r)\left(1-\frac{\xi_2}{2+\xi_2}\cos^2(\theta)\right), \quad (20)$$

with $D_{LL}(r)=\langle(u(x)-u(x+r))^2\rangle$. A more involved relation can be written for the fourth-order moment:

$$S_4(r,\theta)=D_{LLLL}(r)\cos^4(\theta)-3D_{LLNN}(r)\sin^2(2\theta)+D_{NNNN}(r)\sin^2(\theta), \quad (21)$$

where $D_{LLNN}=\langle(v(x)-v(x+r))^2(u(x)-u(x+r))^2\rangle$, and v and u are the components of the velocity field perpendicular and parallel to the x axis, respectively. In general, in the limit $\cos(\theta)\equiv s\to\pm 1$, corresponding to moments of the longitudinal velocity differences $S_n(r,s)\to S_n(r)\cos^n(\theta)$. This means that in this limit $Z(\lambda,r,s)\to Z(\lambda s,r)\equiv Z(\lambda_x,r)$. The generating function can depend only on three variables:

$$\eta_1=r,\quad \eta_2=\frac{\boldsymbol{\lambda}\cdot\mathbf{r}}{r}\equiv\lambda\cos(\theta),\quad \eta_3=\sqrt{\lambda^2-\eta_2^2}.$$

In these variables,

$$Z_t+\left[\partial_{\eta_1}\partial_{\eta_2}+\frac{d-1}{r}\partial_{\eta_2}+\frac{\eta_3}{r}\partial_{\eta_2}\partial_{\eta_3}+\frac{(2-d)\eta_2}{r\eta_3}\partial_{\eta_3}-\frac{\eta_2}{r}\partial^2_{\eta_3}\right]Z=I_f+I_p, \quad (22)$$

where

$$I_p=\lambda_i\langle(\partial_{2,i}p(2)-\partial_{1,i}p(1))e^{\boldsymbol{\lambda}\cdot\mathbf{U}}\rangle \quad (23)$$

and

$$I_f=(\eta_2^2+\eta_3^2)P(1-\cos(k_f r))Z, \quad (24)$$

where, to simplify notation we set $\partial_{i,\alpha}\equiv\partial/\partial x,\alpha$ and $v(i)\equiv v(x_i)$. In two dimensions, the equation for the generating function becomes

$$\left[\partial_{\eta_1}\partial_{\eta_2}+\frac{1}{r}\partial_{\eta_2}+\frac{\eta_3}{r}\frac{\partial^2}{\partial\eta_2\partial\eta_3}-\frac{\eta_2}{r}\frac{\partial^2}{\partial\eta_3^2}-P(\eta_2^2+\eta_3^2)\right]Z=I_p. \quad (25)$$

The generating function can be written as

$$Z=\langle e^{\eta_2\Delta u+\eta_3\Delta v}\rangle, \quad (26)$$

so that any correlation function

$$\langle(\Delta u)^n(\Delta v)^m\rangle=\frac{\partial^n}{\partial\eta_2^n}\frac{\partial^m}{\partial\eta_3^m}Z(\eta_2=\eta_3=0). \quad (27)$$

Neglecting the pressure term I_p and differentiating Eq. (25) once over η_2 we immediately obtain ($d=2$)

$$\frac{d}{dr}rS_2=S_2^t. \quad (28)$$

A second differentiation (again neglecting I_p) gives

$$\frac{1}{r}\frac{d}{dr}rS_3-\frac{2}{r}S_{3t}-2P=0. \quad (29)$$

Combined with Eq. (13), this expression gives

$$\frac{1}{r^3}\frac{d}{dr}r^3S_3-6P=0, \quad (30)$$

which is nothing but the Kolmogorov relation, derived in two dimensions without contributions from the pressure terms. It follows from Eq. (25) that it is reasonable to look for a scaling solution $Z(\eta_2,\eta_3,r)=Z(X_2,X_3)$, where $X_i=\eta_i r^{1/3}$.

III. POLYAKOV'S THEORY OF BURGERS TURBULENCE

The dissipation-generated contributions are $O(\nu\nabla^2\overline{u_iu_j})\neq 0$ in the limit $\nu\to 0$. This is a consequence of the ultraviolet singularity $\nabla^2\overline{u_i(x)u_j(x+r)}\to\infty$ when $r\to 0$, making the theory (the closure problem) extremely difficult. The expression for this ''dissipation anomaly,'' part of the equation for the generating function, was developed by Polyakov for the problem of the one-dimensional Burgers equation stirred by a random force [8]. The theory of two-dimensional turbulence is free from the troubles coming from the ultraviolet (dissipation) singularities. Still, here we review some of the aspects of Polyakov's theory which we believe are of general interest and which will be most helpful below. Polyakov considered a one-dimensional problem [8]

$$u_t+uu_x=f+\nu u_{xx}, \quad (31)$$

where the random force is defined by the correlation function

$$\overline{f(x,t)f(x+r,t')}=\kappa(r)\delta(t-t'). \quad (32)$$

The equation for a generating function, analogous to Eq. (14), is written readily:

$$Z_t+\sum_j\lambda_j\frac{\partial}{\partial\lambda_j}\frac{1}{\lambda_j}\frac{\partial Z}{\partial x_j}=\kappa(r_{ij})\lambda_i\lambda_jZ+D, \quad (33)$$

where

$$D=\nu\lambda_j\langle u''(x_j,t)e^{\lambda_k u(x_k,t)}\rangle. \quad (34)$$

In the limit $r_{ij}\to 0$ the force correlation function $\kappa(r_{ij})=O(1-r_{ij}^2)$, which imposes scaling properties on the velocity correlation functions. In general, the generating function depends on both velocity differences $U_-=\Delta u=u(x_i)-u(x_j)$ and sums $U_+=u(x_i)+u(x_j)$ making the problem very difficult. Defining Galilean invariance as independence of the correlation functions on ''non-universal'' single-point $u^2_{\rm rms}=\overline{u^2}$, Polyakov assumed that if all $|U_-|\ll u_{\rm rms}$ then U_- and U_+ are statistically independent and $\Sigma\lambda_i=0$. In this case (see Ref. [8]), introducing $\mu=\lambda_2-\lambda_1$ and the two-point generating function

$$Z(\mu)=(e^{\mu\Delta u}) \quad (35)$$

the equation for Z reads, in a steady state,

$$\left(\frac{\partial}{\partial\mu}-\frac{2}{\mu}\right)\frac{\partial}{\partial r}Z=-r^2\mu^2 Z+D, \tag{36}$$

where

$$D=\mu\nu((u''(x+r)-u''(x))e^{\mu\Delta u}). \tag{37}$$

It is clear that the $O(r^2)$ forcing term imposes the scaling variable $\xi=\mu r$ and $Z=F(\mu r)$, where F is a solution of the following equation:

$$\xi F''-F'+\xi^2 F=D \tag{38}$$

The problem is in evaluation of the dissipation contribution D.

At first glance one can attempt to neglect D and solve the resulting equation. This is not so simple, however. The Laplace transform of Eq. (38) gives an equation for the probability density $P=(1/r)\Phi(U/r)\equiv(1/r)\Phi(X)$:

$$\Phi''+X^2\Phi'+3X\Phi=0.$$

Introducing

$$\Phi=\exp\left(-\frac{X^3}{6}\right)\Psi \tag{39}$$

gives

$$\Psi''=\left(\frac{X^4}{4}-2X\right)\Psi, \tag{40}$$

which is the stationary Schrödinger equation for a particle in a potential $U(X)=X^4/4-2X$, not having any positive everywhere solutions $P(X)$. This difficulty can be understood readily since Eq. (40) corresponds to a particle having the ground-state energy $E_0=0$. It can be shown that only when the linear in X contribution to potential $U(X)$ is modified, so that

$$U(X)=\frac{X^4}{4}-\frac{3}{2}X,$$

the equation

$$\Phi''(X)=U(X)\Phi(X)$$

does give an everywhere positive solution $P(X)$.

The positivity of the probability density is a severe constraint on a possible solution of the equation of motion. That is where the dissipation contribution D comes to the rescue. Polyakov proposed a self-consistent conjecture about the structure of the dissipation term

$$D=\left(\frac{b}{\mu}+a\right)Z \tag{41}$$

modifying the potential in the Schrödinger equation with the coefficients b and a chosen to produce the zero-energy ground state corresponding to a positive PDF. According to Ref. [8] this expression is the only one satisfying the Galilean invariance of the small-scale dynamics. The fact that the one-dimensional or multidimensional advection contributions to the equation for the generating function do not lead to positive solutions for the PDF is a general phenomenon (see below). The importance of Polyakov's theory is, among other things, in realization that the dynamic closures for the remaining terms must remove this problem. This dramatically narrows the allowed classes of closures. Thus, the expressions for D or the pressure terms (see below), combined with advective contributions to equation for Z, can be correct if and only if they lead to positive solutions for the PDF's in the entire range where $|\Delta u|\ll u_{\rm rms}$ and $r\ll L_i$.

IV. PHYSICAL CONSIDERATIONS

The problem of two-dimensional turbulence is simplified by the fact that the dissipation contributions are irrelevant on the scales $l\gg l_f$ we are interested in. Moreover, since $u_{\rm rms}$ grows with time, the statistically steady small-scale velocity differences $U_-=\Delta u$ with $r\ll L(t)$ must be decoupled from U_+ in Eq. (25). This means that the terms

$$\overline{(\Delta u)^n(\Delta u)^m} \tag{42}$$

can enter the equation for $P(\Delta u,r)$, while the ones involving

$$\overline{(\Delta u)^n(\Delta v)^m U_+^p} \tag{43}$$

cannot. In principle, it can happen that the U_-U_+ correlation functions can sum up into something time independent. However, at present we discard this bizarre possibility.

Next, the pressure gradients

$$\nabla p(x+r)-\nabla p(x) \tag{44}$$

appearing in Eqs. (22)–(24) for Z involve integrals over entire space. It is clear that, if a steady state exists, the large scale contribution to the pressure integrals, depending on $L=L(t)$, cannot contribute to the small-scale steady-state dynamics described by Eq. (25). That is why the pressure contributions to I_p Eq. (23) must depend exclusively on the local scale r. This leads us to an assumption that the pressure gradients in Eq. (23) are local in the sense that they can be expressed in terms of the velocity field at the points x and $x+r$. The application of these considerations are presented below.

The theory of Burgers turbulence dealt with the "universal" part of the dynamics, i.e., with the moments of velocity difference S_n with $n<1$. The theory of two-dimensional turbulence that we are interested in must produce the moments with $n<\infty$, and that is why the algebtaic expressions for the PDF's, characteristic of Burgers dynamics, are irrelevant. In addition, we expect the small-scale dynamics in two dimensional to be independent of the forcing function. This makes this problem very different.

V. TRANSVERSE STRUCTURE FUNCTIONS

Unlike the probability density function for the longitudinal velocity differences $P(\Delta u,r)$, the transverse velocity difference probability density is symmetric, i.e., $P(\Delta v,r)=P(-\Delta v,r)$. We are interested in Eq. (25) in the limit $\eta_2\to 0$. Let us first discuss some of the general properties of incompressible turbulence. Consider the forcing function

$$f(x,y)=(f_x(x,y),0).$$

In this case Eq. (25) is

$$\left[\partial_{\eta_1}\partial_{\eta_2}+\frac{1}{r}\partial_{\eta_2}+\frac{\eta_3}{r}\frac{\partial^2}{\partial_{\eta_2}\partial_{\eta_3}}-\frac{\eta_2}{r}\frac{\partial^2}{\partial\eta_3^2}-P\,\eta_2^2\right]Z=I_p\,. \tag{45}$$

Then setting $\eta_2=0$ removes all information about the forcing function from the equation of motion. Based on our general intuition and numerical data, we know that two flows stirred by a one or two-component (statistically isotropic) forcing function are identical at the scales $l\gg l_f$, provided the total fluxes generated by these forcing functions are equal. This happens due to pressure terms

$$\Delta p=-\nabla_i\nabla_j v_i v_j$$

effectively mixing various components of the velocity field. This universality, i.e., independence of the small-scale turbulence on the symmetries of the forcing, enables us to write an expression for the I_p contribution to Eq. (25).

According to considerations presented in a previous section, the pressure gradients in Eq. (25) are local and their dynamic role is in mixing various components of velocity field. Thus the only contribution to I_p, not vanishing in the limit $\eta_2\to 0$, can be estimated as

$$b\,\frac{\eta_3}{r}\langle \delta u\Delta v e^{\eta_2\Delta u+\eta_3\Delta v}\rangle=b\,\frac{\eta_3}{r}\frac{\partial}{\partial\eta_2}\langle \Delta v e^{\eta_2\Delta u+\eta_3\Delta v}\rangle. \tag{46}$$

Using a theorem (see Frisch [7], for example) that for the random Gaussian process ξ (see below)

$$\langle \xi F(\xi)\rangle=\overline{\xi^2}\left\langle\frac{\partial F(\xi)}{\partial\xi}\right\rangle, \tag{47}$$

we derive, in the limit $\eta_2\to 0$,

$$I_p\approx b\,\eta_3^2\frac{\overline{(\Delta v)^2}}{r}\frac{\partial Z_3}{\partial\eta_2}. \tag{48}$$

Substituting this into Eq. (25) and integrating over η_2 gives, in the limit $\eta_2\to 0$,

$$\frac{\partial Z_3}{\partial r}+\frac{Z_3}{r}+\frac{\eta_3}{r}\frac{\partial Z_3}{\partial\eta_3}-\frac{\gamma P}{(Pr)^{1/3}}\eta_3^2 Z_3+\Omega(\eta_3)=\Gamma(\eta_3) \tag{49}$$

where γ is undetermined parameter and an arbitrary function

$$\Gamma(\eta_3)=Z_3/r+\Omega(\eta_3),$$

with

$$-\Omega(\eta_3)=\lim_{\eta_2\to 0}\eta_3^2\int Z(\eta_2,\eta_3,r)d\eta_2$$

is chosen to satisfy a trivial constraint $Z_3(\eta_3=0,r)=1$ and the above mentioned universality.

This gives

$$\frac{\partial Z_3}{\partial r}+\frac{\eta_3}{r}\frac{\partial Z_3}{\partial\eta_3}-\frac{\gamma P}{(Pr)^{1/3}}\eta_3^2 Z_3=0, \tag{50}$$

where $Z_3=Z(\eta_2=0,\eta_3)$. This equation is invariant under $\eta_3\to-\eta_3$ transformation. It is important that the $O(\eta_3^2)$ contribution to Eq. (50) corresponds to the pressure term but not from the forcing present in the original equation (25). Seeking a solution to this equation in a scaling form (we set $P=1$ for now) $Z_3(\eta_3,r)=Z(\eta_3 r^{1/3})\equiv Z(X)$ gives

$$\frac{4X}{3}Z_X=\gamma X^2 Z \tag{51}$$

and

$$Z=\exp\left(\frac{3\gamma}{8}\eta_3^2 r^{2/3}\right). \tag{52}$$

This generating function corresponds to the Gaussian distribution of transverse velocity differences $P(\Delta v)$ with the second-order structure function

$$S_2^t(r)=\overline{(\Delta v)^2}=\frac{3\gamma}{4}r^{2/3} \tag{53}$$

Equation (50) corresponds to a one-dimensional linear Langevin equation for ''velocity field'' $V=v/(Pr)^{1/3}$,

$$v_r(x)=-v(x)+\phi(x,\tau), \tag{54}$$

where $\tau\propto tr^{-2/3}P^{1/3}$, and the nonlocal Gaussian ''universal'' forcing $\phi(x,\tau)$, generated by the nonlinearity of the original equation, is defined by the correlation function

$$\overline{\phi(k,\tau)\phi(k',\tau')}\propto\delta(k+k')\delta(\tau-\tau'). \tag{55}$$

The generating function for the field V is

$$z=\langle e^{XV}\rangle.$$

Since $\tau\propto tr^{-2/3}$ and $V\propto vr^{-1/3}$, this equation is strongly nonlocal. It becomes local, however in the wave-number space. This will be discussed later.

Now we can attempt to justify relation (46). According to Eq. (23), and taking into account that the x axis is parallel to the displacement r in the limit $\eta_2\to 0$,

$$I_p\approx\eta_3((\partial_y p(0)-\partial_{y'}p(r))\exp(\eta_3\Delta v+\eta_2\Delta u)),$$

where

$$\partial_y p(0)-\partial_{y'}p(r)=\int k_y(1-e^{ik_x r})\left[\frac{k_x^2}{k^2}u(q)u(k-q)+\frac{k_y^2}{k^2}v(q)v(k-q)+\frac{k_xk_y}{k^2}u(q)\times v(k-q)\right]d^2k d^2q,$$

and the exponent is expressed simply as

$$e^{\eta_3\Delta v+\eta_2\Delta u}=\exp\left[\eta_3\int(1-e^{iQ_xr})v(Q)d^2Q+\eta_2\int(1-e^{iQ_xr})u(Q)d^2Q\right].$$

It will be come clear below that transverse velocity differences Δv obey Gaussian statistics, and the longitudinal ones Δu are very close to Gaussian. Then, substituting the above expressions into I_p and expanding the exponent we generate an infinite series involving various products of $u(q)$'s and $v(q)$'s. In the case of the incompressible, statistically isotropic Gaussian velocity field we are dealing with, these products are split into pairs:

$$\langle v_i(q)v_j(Q)\rangle\propto q^{-8/3}\left(\delta_{ij}-\frac{q_iq_j}{q^2}\right)\delta(q+Q).$$

The k_y integration is carried over the interval $-\infty<k<\infty$, and in the isotropic case we are dealing with the only nonzero terms are those involving even powers of k_y. These terms are generated by the expansion of

$$e^{\eta_2\Delta u}.$$

However, being $O(\eta_2)$, they disappear in the limit $\eta_2\to0$. Thus

$$I_p=\eta_3\int d^2kd^2qk_y(1-e^{ik_xr})\frac{k_xk_y}{k^2}\left\langle u(q)v(k-q)\exp\left(\eta_3\int(1-e^{iQ_xr})v(Q)d^2Q+\eta_2\int(1-e^{iQ_xr})u(Q)d^2Q\right)\right\rangle,$$

where the $O(\eta_2)$ contribution to the exponent is temporarily kept to make the transformation

$$\Delta ue^{\eta_2\Delta u}=\frac{\partial e^{\eta_2\Delta u}}{\partial\eta_2}$$

to Eq. (46) possible. Only after that do we set $\eta_2=0$. This proves that the only contribution to the equation for the probability density function comes from the $O(\Delta u\Delta v)$ mixing components, involved in the pressure gradients. This relation justifies estimate (46).

VI. LONGITUDINAL VELOCITY DIFFERENCES

The remarkable fact that in the limit $\eta_2\to0$ all contributions to Eq. (25) contain $\partial/\partial_{\eta2}$ enables a separation of the variables: integrating the resulting equation over η_2 gives the closed equation for $Z_3(\eta_3)$. The corresponding dynamic equation is linear, meaning that transverse velocity fluctuations do not directly contribute to the energy transfer between different scales. This effect is possible for an arbitrary value of η_2 only in two dimensions, where the $O((d-2)(\partial/\partial\eta_3))$ enstrophy production term in Eq. (22), not containing $\partial/\partial\eta_2$, is equal to zero. This simplification, combined with the locality of the pressure-gradient effects, allowed us to derive a closed-form expression for Z_3.

The role of pressure in the dynamics of transverse components of velocity field is mainly restricted to control of the ''energy redistribution'' necessary for generation of an isotropic and incompressible velocity field. The longitudinal field dynamics are much more involved. The advection (pressure excluding) part of the nonlinearity tends to produce large gradients of velocity field (''shock generation'' using the Burgers equation phenomenology), manifesting itself in creation of a constant energy flux in the wave-number space. Pressure is the only factor preventing shock formation.

Since we are interested in the longitudinal correlation functions, we set $\eta_3=0$. Then, defining $S_2^t=((\Delta v)^2)\approx A_2^t(Pr)^{2/3}$, and setting $P=1$ the term in Eq. (25) can be rewritten as

$$\frac{\eta_2}{r}\frac{\partial^2Z}{\partial\eta_3^2}=\frac{\eta_2}{r}\langle(\Delta v)^2e^{\eta_2\Delta u}\rangle\approx\frac{\eta_2A_2^t}{r^{1/3}}Z_2+O(\eta_2^2;\eta_3^2;\eta_2^2\eta_3). \tag{56}$$

The last relation is accurate since, substituting this into Eq. (25), differentiating once over η_2, and setting both $\eta_3=\eta_2=0$, gives

$$\frac{1}{r}\frac{\partial}{\partial r}rS_2-\frac{A_2^t}{r^{1/3}}=\frac{\partial I_p(0,0)}{\partial\eta_2}. \tag{57}$$

Since $S_2(r)=A_2r^{2/3}$, this equation gives

$$\frac{5}{3}A_2-A_2^t=r^{1/3}\frac{\partial I_p(0,0)}{\partial\eta_2}, \tag{58}$$

which, according to Eq. (12) is exact since $\partial I_p(0,0)/\partial\eta_2=0$ (see below).

Let us consider some general properties of the pressure term I_p in the limit $\eta_3\to0$. We have

$$I_p\approx\eta_2\left\langle\left(\frac{\partial p(2)}{\partial x_2}-\frac{\partial p(1)}{\partial x_1}\right)\exp(\eta_2\Delta u+\eta_3\Delta v)\right\rangle. \tag{59}$$

Expanding the exponent and recalling that for isotropic and incompressible turbulence $\overline{\Delta u}=\overline{\Delta v}=0$ and $\overline{p(x)v_i(x')}=0$, we conclude that

$$I_p\approx\eta_2\left\langle\left(\frac{\partial p(2)}{\partial x_2}-\frac{\partial p(1)}{\partial x_1}\right)(\eta_2\Delta u+\eta_3\Delta v)^2+\cdots\right\rangle=O(\alpha\eta_2^3+\beta\eta_2^2\eta_3+\cdots). \tag{60}$$

It is clear that relation (48), derived above for the case of Gaussian statistics, satisfied this general property of the flow. Thus when $\eta_3\to0$, we approximate

$$I_p\approx cr^{1/3}\eta_2^3Z+G, \tag{61}$$

where c is an as yet undetermined constant, and G denotes the contributions to I_p, properly modifying numerical coefficients in Eq. (25). The presence of the $O(\eta_2^3)$ distinguishes this equation from the one for the transverse PDF considered in Sec. V. There the assumed role of pressure was limited to the mixing of various components of velocity field. That is why all we accounted for was $O(\Delta v\Delta u)$ contributions to pressure. Here, in addition, we also consider $O(\eta_2^3)$ contributions, responsible for prevention of the shock formation. The resulting equation is

$$\frac{1}{Pr^3}\frac{\partial^2}{\partial\eta_2\partial r}r^3Z_2-\frac{11}{5(Pr)^{1/3}}A_2^t\eta_2Z_2-3\eta_2^2Z_2 -c(Pr)^{1/3}\eta_2^3Z_2=0. \tag{62}$$

Differentiating Eq. (62) once over η_2, and setting $\eta_2=0$, gives $A_2^t=5/3A_2$, in accord with the general relation (12) for $d=2$. The Laplace transform of Eq. (62) gives equation for the probability density $P(\Delta u,r)$:

$$c(Pr)^{1/3}P_{UUU}-3P_{UU}+\frac{1}{Pr^3}\frac{\partial}{\partial r}r^3UP+\frac{11A_2^t}{5(Pr)^{1/3}}P_U=0. \tag{63}$$

Seeking a solution in a scaling form (the parameter c will be determined below),

$$P(U,r)=\frac{1}{r^{1/3}}F\left(\frac{U}{(Pr)^{1/3}}\right), \tag{64}$$

we obtain, again for simplicity setting $P=1$,

$$cF_{xxx}-3F_{xx}+\left(b-\frac{x^2}{3}\right)F_x+\frac{8}{3}xF=0, \tag{65}$$

where $b=11/5A_2^t=11/3A_2$. All but one term in Eq. (65) changes sign when $x\to -x$. The $O(F_{xx})$ symmetry-breaking contribution is necessary for the existence of the nonzero energy flux. Assuming for the time being, in accord with numerical and physical experiments, that the flux is small [see relation (2)], we first neglect the $O(F_{xx})$ contribution. The equation is

$$cF^0_{xxx}+\left(b-\frac{x^2}{3}\right)F^0_x+\frac{8}{3}xF^0=0, \tag{66}$$

with solution

$$F^0=e^{x^2/2A_2}, \tag{67}$$

where $c=A_2^2/3$. If $A_2\gg 1$, then the neglected $F_{xx}=O(1/A_2)$ term is numerically small. This means that the odd-order moments, computed with the PDF, which is a solution of Eq. (65), must be small in a sense defined by relation (2). At the same time the even-order moments must be close to the gaussian ones.

An analytic solution of Eq. (65) is difficult. However, one can evaluate all moments $S_n/r^{n/3}=A_n$ in terms of only one parameter A_2:

$$S_{n+1}=-\frac{3}{n+10}\left(-\frac{A_2^2}{3}n(n-1)(n-2)S_{n-3}-3n(n-1)S_{n-2}-\frac{11}{3}A_2nS_{n-1}\right). \tag{68}$$

This relation gives $A_1=0$, $A_3=\frac{3}{2}$, $A_4=3$, $A_5=12.43A_2$, $A_6=15A_2^3+36$, $A_7=37.71A_4$, etc. These numbers can be tested in numerical experiments. The one-loop renormalized perturbation expansions give $A_2\approx 10$, while numerical simulations are consistent with $A_2\approx 12$. Keeping these numbers in mind, it follows from Eq. (68) that accurate measurements of the odd-order moments are the only way to verify predictions of the present theory. The deviations of the even-order moments from the Gaussian ones are too small to be detected by both physical and numerical experiments. It can be checked that the ratios

$$s_{2n+1}=\frac{S_{2n+1}}{S_{2n}^{(2n+1)/2n}}$$

vary in the interval 0.04–0.1 for $2<n<10$ and $A_2\approx 10$. With $A_2\approx 12$, these numbers are even smaller.

VII. SUMMARY AND CONCLUSIONS

The experimentally observed Gaussian or very nearly Gaussian statistics of transverse velocity differences was extremely puzzling since, at first glance, this is incompatible with the nontrivial Kolmogorov scaling resulting from the strong nonlinearity of the problem. The most surprising and interesting result, derived in this paper, is that due to the symmetries of the problem the equation, governing the probability density function of transverse velocity differences, has one derivative less than the one corresponding to the longitudinal differences. This means, in turn, that transverse components of the velocity field are governed by a nonlocal linear equation, driven by a universal, nonlocal, solution-dependent Gaussian force. This reduction, resembling the supersymmetry effects in field theory, is surprising if not miraculous. The nonlocal equation in physical space, obtained above, corresponds to the Langevin equation in the Fourier space:

$$v_t(k)+c_\nu P^{1/3}k^{2/3}v=f_R(k,t), \tag{69}$$

where c_ν is an amplitude of ''effective'' (turbulent) viscosity, and

$$\overline{f_R(k,t)f_R(k',t')}\propto k^{-1}\delta(k+k')\delta(t-t') \tag{70}$$

is the force covariance used in Refs. [9], [10] in the renormalization group treatments of fluid turbulence.

The irrelevance of the dissipation terms in two-dimensional turbulence makes the problem much more tractable than its three-dimensional counterpart. Still, in order to close equations for the probability density of the velocity field, one needs an expression for the pressure contributions. The situation is even more simplified by the fact that the large-scale-dominated single-point variables are time dependent and must decouple from the steady-state small-scale dynamics. That is why one can use an assumption about the locality of the pressure gradient effects, leaving only the mixing $O(\Delta u\Delta v)$ contributions to the two-point pressure difference. It can be tested by a mere accounting that all other contributions to the expression for I_p involve one or more U_+'s and lead to a time-dependent result. This means that they must disappear from the steady-state equations (25) and (45). The range of possible models for pressure is narrowed by a few dynamic and kinematic constraints, and by the fact that the resulting equation must give a non-negative solution everywhere. A simple calculation shows that the model for the pressure gradient terms, introduced in this paper, is consistent with the derived Gaussian statistics.

The equations for PDF is of longitudinal velocity differences do not correspond to linear dynamics. Still, the derived solution only slightly deviates from Gaussian. This is possible due to the relative smallness of the energy flux in two dimensions.

The results presented here seem to agree with both physical and numerical experiments. The nearly Gaussian statistics obtained above justifies various one-loop renormalized perturbation expansions giving $A_2 \approx 10-12$. Using this number we realize that it is extremely difficult to detect deviations from Gaussian statistics experimentally. Still, some fine details of the present theory, related to the pressure gradient-velocity correlation functions can be tested numerically. In addition, measurements of a few odd-order moments can shed some light on the validity of the present theory.

The equations and solution presented here leave one question unanswered: are these the solutions or not? Our experience with the Burgers and two-dimensional Navier-Stokes equations teach us that it is very difficult to find a self-consistent closure leading to the positive solution for the PDF's. Stretching this statement a bit, we feel that a closure, satisfying dynamic constraints and leading to a plausable solution, has a good chance to be correct.

The absence of intermittency in a steady-state developing inertial range, discovered in two-dimensional turbulence [2–4] seems to be a general phenomenon observed in drift-wave turbulence [11] and in a one-dimensional model of a passive scalar advected by a compressible velocity field [12]. These observations support our understanding of intermittency as a phenomenon originating from the interaction of large- and small-scale velocity fluctuations. In a developing statistically steady inertial range, where the integral scale is strongly time dependent, these interactions must be small for the small-scale steady state to exist. At later stages finite size effects, destroying the time independence of the small-scale dynamics, lead to formation of coherent structures and new dynamic phenomena which are beyond the scope of the present theory.

Note added in proof. In a recent paper Bofetta, Celatti, and Vergassola [13] reported the results of very accurate numerical simulations of two-dimensional turbulence generated by the random force. The computed odd-order moments S_5 and S_7 were very close to the ones obtained in the present study from relation (68). No deviations from the Gaussian statistics of transverse velocity differences were detected. Moreover, the measured PDF of the longitudinal differences could be represented as a sum: $P(\Delta u) = P_s(\Delta u) + P_a(\Delta u)$ where $P_s(x) = P_s(-x)$ and $P_a(x) = -P_a(-x)$ with $P_s(x)$ indistinguishable from the Gaussian. The same feature can be derived from Eq. (68).

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